

# CONVERGENCE OF MEASURES UNDER DIAGONAL ACTIONS ON HOMOGENEOUS SPACES

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**ABSTRACT.** Let  $\lambda$  be a probability measure on  $\mathbb{T}^{n-1}$  where  $n = 2$  or  $3$ . Suppose  $\lambda$  is invariant, ergodic and has positive entropy with respect to the linear transformation defined by a hyperbolic matrix. We get a measure  $\mu$  on  $SL_n(\mathbb{Z}) \backslash SL_n(\mathbb{R})$  by putting  $\lambda$  on some unstable horospherical orbit of the right translation of  $a_t = \text{diag}(e^t, \dots, e^t, e^{-(n-1)t})$  ( $t > 0$ ). We prove that if the average of  $\mu$  with respect to the flow  $a_t$  has a limit, then it must be a scalar multiple of the probability Haar measure. As an application we show that if the entropy of  $\lambda$  is large, then Dirichlet's theorem is not improvable  $\lambda$  almost surely.

## 1. INTRODUCTION

For any positive integer  $p$ , let  $\times p$  be the map of  $[0, 1] = \mathbb{Z} \backslash \mathbb{R}$  to itself which sends  $x \bmod \mathbb{Z}$  to  $px \bmod \mathbb{Z}$ . Let  $q$  be another positive integer such that  $p$  and  $q$  are not powers of any integer, then it is conjectured by Furstenberg that the only non-atomic  $\times p$  and  $\times q$  ergodic probability measure on  $[0, 1]$  is the Lebesgue measure. So far the best results known are obtained by Rudolph [13] and Johnson [7] under the assumption of positive entropy with respect to the  $\times p$  map. Using their measure rigidity results they proved in [8] that the average of the  $\times q$  orbit of any  $\times p$  ergodic probability measure with positive entropy converges to the Lebesgue measure. Here and hereafter all the convergence of measures on some locally compact topological space  $X$  are under the weak\* topology of Radon measures  $\mathcal{M}(X)$ .

Recently, the measure rigidity on torus is generalized to homogeneous spaces by Einsieder, Katok and Lindenstrauss, see [4], [10] and [5] for details. Using these new results we can extend the convergence of measures in [8] to homogeneous spaces.

For a lattice  $\Gamma$  of a locally compact group  $G$ , we use  $m_{\Gamma \backslash G}$  to denote the unique probability Haar measure on  $\Gamma \backslash G$  and use  $\vartheta_{\Gamma \backslash G}$  to denote the element  $\Gamma$  in  $\Gamma \backslash G$ . An element  $g \in G$  also stands for the map on  $\Gamma \backslash G$  which sends  $x \in \Gamma \backslash G$  to  $xg$ . Let  $\mathbb{R} \rightarrow G$  be a continuous homomorphism and  $a_t$  be the image of  $t \in \mathbb{R}$ . We consider the one-parameter subgroup  $\{a_t\}$  as a flow on  $\Gamma \backslash G$  through right translation. For a Radon measure  $\mu$  on  $\Gamma \backslash G$ , let  $a_t \mu$

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be the pushforward of  $\mu$  under the map  $a_t$ , that is  $a_t\mu(C) = \mu(Ca_t^{-1})$  for any Borel measurable subset  $C \subset \Gamma \backslash G$ . We say  $\mu$  has non-escape of mass with respect to the flow  $a_t$ , if any limit of

$$(1.1) \quad \left\{ \int_0^T a_t\mu \, dt : T \geq 0 \right\}$$

is a probability measure.

We consider Euclidean space  $\mathbb{R}^{n-1}$  as the space of row vectors. For any  $\mathbf{s} \in \mathbb{R}^{n-1}$ , take

$$(1.2) \quad u(\mathbf{s}) = \begin{pmatrix} \text{Id}_{n-1} & 0 \\ \mathbf{s} & 1 \end{pmatrix} \in SL_n(\mathbb{R}).$$

**Theorem 1.1.** *Let  $\lambda$  be the probability Cantor measure on  $[0, 1]$ . Suppose  $a_t = \text{diag}(e^t, e^{-t}) \in SL_2(\mathbb{R})$  and  $Y = SL_2(\mathbb{Z}) \backslash SL_2(\mathbb{R})$ . Then for any  $f \in C_c(Y)$ ,*

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \int_0^1 f(\vartheta_Y u(s) a_t) \, d\lambda(s) \, dt = \int_Y f \, dm_Y.$$

The above theorem strengthens Theorem 1.6 of [2] where it is proved that for  $\lambda$  almost every  $y \in Y$  the orbit  $ya_t$  with  $t \in \mathbb{R}$  is dense. A general version of Theorem 1.1 will be proved in Section 5. The idea is to lift the flow  $a_t$  and the measure on  $Y$  to an irreducible quotient of product of real and  $p$ -adic Lie groups which follows [2], then we prove equidistribution there.

To state this result, we need to introduce some notation. Let  $S = \{\infty, p_1, \dots, p_l\}$  where  $p_i$  is prime and  $l > 0$ . Take  $\mathbb{Q}_S = \prod_{v \in S} \mathbb{Q}_v$  where  $\mathbb{Q}_v$  is the completion of  $\mathbb{Q}$  at the place  $v$  and integer  $n = p_1^{\sigma_1} \cdots p_l^{\sigma_l}$  where  $\sigma_i > 0$ . For  $\mathbf{s} \in \mathbb{Q}_S$ , we define  $u(\mathbf{s}) \in SL_2(\mathbb{Q}_S)$  as in (1.2) with  $n = 2$ . In this paper we consider elements of  $GL_2$  acts on quotients of  $PGL_2$  through the natural projection map and use elements of  $GL_2$  to represent their image in  $PGL_2$ .

**Theorem 1.2.** *Let  $\mu$  be a probability measure on*

$$X = PGL_2(\mathbb{Z}[1/p_1, \dots, 1/p_l]) \backslash PGL_2(\mathbb{Q}_S)$$

*supported on  $\vartheta_X u(\mathbb{Q}_S)$ . Suppose  $\mu$  is invariant, ergodic and has positive entropy with respect to the right translation of  $\text{diag}(n, 1)^{l+1}$ . If  $\mu$  has non-escape of mass with respect to the flow  $a_t = \text{diag}(e^t, e^{-t})$  of the  $PGL_2(\mathbb{R})$  factor, then  $\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T a_t\mu \, dt$  is an  $H$  invariant homogeneous measure.*

Theorem 1.2 is based on the Theorem 3.1, which shows that all the ergodic components of the limit of (1.1) with respect to the flow  $a_t$  have positive entropy. Then we apply measure rigidity in Lindenstrauss [10]. In fact Theorem 3.1 is a general result about entropy and can also be applied to the homogeneous space  $Z = SL_3(\mathbb{Z}) \backslash SL_3(\mathbb{R})$  even in the case that there is some mass left.

Elements of  $SL(2, \mathbb{Z})$  acts naturally on  $\mathbb{T}^2 = \mathbb{Z}^2 \backslash \mathbb{R}^2$  from right by matrix multiplications. We can put a probability measure  $\lambda$  of  $\mathbb{T}^2$  on  $Z$  to get a measure  $\mu$  in the following way: for any  $f \in C_c(Z)$ ,

$$(1.3) \quad \int_{\mathbb{T}^2} f(\vartheta_Z u(s)) d\lambda(s) = \int_Z f d\mu.$$

**Theorem 1.3.** *Let  $\gamma \in SL_2(\mathbb{Z})$  be a hyperbolic matrix and let  $\lambda$  be an  $\gamma$ -invariant and ergodic probability measure on  $\mathbb{T}^2$  with positive entropy. Suppose  $a_t = \text{diag}(e^t, e^t, e^{-2t})$  and  $\mu$  is the measure on  $Z$  induced from  $\lambda$  by (1.3). If for a sequence of real numbers  $(T_n)$  with  $T_n \rightarrow \infty$ , the limit of*

$$(1.4) \quad \mu_n = \frac{1}{T_n} \int_0^{T_n} a_t \mu dt$$

*exists, then  $\lim \mu_n = c\mu_Z$  where  $0 \leq c \leq 1$ .*

We remark here that a large class of measures on  $[0, 1]^2$  such as friendly measures of [9] satisfy  $c = 1$ . Also by a theorem of [3], if the entropy of  $\lambda$  with respect to  $\gamma$  is large, then  $c > 0$ . The equidistribution result of Theorem 1.3 has the following number theoretical applications.

**Theorem 1.4.** *Let  $\gamma \in SL_2(\mathbb{Z})$  be a hyperbolic matrix. Suppose  $\lambda$  is a  $\gamma$ -invariant and ergodic probability measure on  $\mathbb{T}^2$  such that  $h_\lambda(\gamma) = c h_m(\gamma)$  where  $m$  is the Lebesgue measure and  $c > \frac{2}{3}$ , then Dirichlet's theorem can not be improved (see Section 7)  $\lambda$  almost surely.*

*Remark 1.5.* Here the number  $\frac{2}{3}$  comes from the mass estimate in [3]. For  $c \leq \frac{2}{3}$  we don't know whether there is always a sequence  $(T_n)$  so that the limit in Theorem 1.3 is non-zero.

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## 2. PRELIMINARIES: NOTATION AND LEAF-WISE MEASURES

Let  $L$  be a product of semisimple algebraic groups, that is  $L = G_1 \times \cdots \times G_l$  where  $G_i$  is a semisimple linear algebraic group over a local field of characteristic zero. Let  $G = SL_n(\mathbb{R}) \times L$  where  $L$  could be trivial and  $n \geq 2$ . Here are some conventions that will be used throughout the paper. We use  $\text{Id}$  to denote the identity element of various groups according to the context. We consider  $SL_n(\mathbb{R})$ ,  $G_i$  and their products also as the corresponding subgroups of  $G$ , and they are called factors of  $G$ .

Suppose a factor group  $H$  of  $G$  is an algebraic group over the local field  $k$  with a fixed absolute value. Let  $g \in H$  be a  $k$ -split semisimple element such that the adjoint action  $\text{Ad}_g$  of  $g$  on the Lie algebra of  $H$  satisfies

- 1 is the only eigenvalue of norm 1;
- No two different eigenvalues have the same norm.

Then following Margulis and Tomanov [12] we say  $g$  is from the class  $\mathcal{A}$ . We remark here that our concept of class  $\mathcal{A}$  element is a little bit different from that of [12] but follows that of [6]. We say  $a \in G$  is an element from the class  $\mathcal{A}$  if all of its components are from the class  $\mathcal{A}$ .

We fix a non-neural element  $a \in G$  from the class  $\mathcal{A}$ . The unstable horospherical subgroup of  $a$  is

$$G^+ = \{g \in G : a^{-n}ga^n \rightarrow e \text{ as } n \rightarrow -\infty\}.$$

We say a subgroup  $U$  of  $G$  is a one-dimensional unipotent subgroup if  $U$  is an algebraic subgroup of a factor of  $G$  and  $U$  is isomorphic to the base field considered as an additive algebraic group. Let  $U$  be a one dimensional subgroup of  $G^+$  normalized by  $a$ . The norm on  $k$  induce a metric  $d^U$  on  $U$ . For  $u \in U$ , let  $B_r^U(u)$  or  $B_r^U$  if  $u$  is the identity stand for the ball of radius  $r$  in  $U$  centered at  $u$ .

Let  $X$  be a locally compact topological space and  $G$  acts on  $X$  from right. Next we review leaf-wise measures along  $U$  foliations for an arbitrary  $a$ -invariant finite positive measure  $\nu$  on  $X$ . More details can be found in [10] and [6]. Let  $\mathcal{M}(U)$  be the set of Radon measures on  $U$ . There is a measurable map  $X \rightarrow \mathcal{M}(U)$  and the image of  $x \in X$  is denoted by  $\nu_x^U$ . The measure  $\nu_x^U$  is called leaf-wise measure along  $U$  foliations and is normalized so that  $\nu_x^U(B_1^U) = 1$ . The entropy contribution of  $U$  with respect to  $\nu$  is an  $a$ -invariant measurable function on  $X$  defined by

$$(2.1) \quad h_\nu(a, U)(x) = \lim_{n \rightarrow \infty} \frac{\log \nu_x^U(a^{-n}B_1^U a^n)}{n}.$$

Zero entropy contribution is characterized as follows:

**Theorem 2.1** (Theorem 7.6 of [6]). *For  $\nu$  almost every  $x \in X$ ,  $h_\nu(a, U)(x) = 0$  if and only if  $\nu_x^U$  is trivial, that is an atomic measure concentrated on the identity.*

Leaf-wise measures are constructed by pasting various conditional measures with respect to countably generated sigma-rings whose atoms are orbits of open bounded subsets of  $U$ . For us sigma-ring is a collection of Borel subsets of  $X$  which are closed under countable unions and set differences. For example, given a finite partition  $\mathcal{P}$  of a set  $D$ , then arbitrary unions of elements of  $\mathcal{P}$  is a sigma-ring with maximal element  $D$ . In this paper we will not distinguish between a finite partition and the sigma-ring it generates. Let  $\mathcal{A}$  and  $\mathcal{B}$  be two sigma-rings, then  $\mathcal{A} \vee \mathcal{B}$  stands for the smallest sigma-ring containing both of them.

Let  $D$  be a measurable subset of  $X$  and let  $\mathcal{A}$  be a countably generated sigma-ring with maximal element  $D$ . For each  $x \in D$  the atom of  $x$  with respect to  $\mathcal{A}$  is

$$[x]_{\mathcal{A}} = \bigcap_{x \in C \in \mathcal{A}} C.$$

Let  $\mathcal{M}(X)$  be the set of Radon measures on  $X$ . We can assign measurably for each  $x \in D$  a probability measure  $\nu_x^{\mathcal{A}} \in \mathcal{M}(X)$  supported on  $[x]_{\mathcal{A}}$ . Here

$\nu_x^{\mathcal{A}}$  is called the conditional measure of  $\nu|_D$  with respect to  $\mathcal{A}$ . If  $\mathcal{A}$  is a finite partition of  $D$ , conditional measure is uniquely determined by

$$\nu_x^{\mathcal{A}}(C)\nu([x]_{\mathcal{A}}) = \nu([x]_{\mathcal{A}} \cap C)$$

for any Borel subset  $C$  of  $X$ .

The sigma-ring  $\mathcal{A}$  is said to be subordinate to  $U$  on  $D$  if for  $\nu$  almost every  $x \in D$ , we have  $[x]_{\mathcal{A}} = U_x \cdot x$  where  $U_x$  is an open subset of  $U$  and there exists  $\delta > 0$  such that

$$B_{\delta}^U \subset U_x \subset B_{\delta-1}^U.$$

If  $\mathcal{A}$  is subordinate to  $U$ , then leaf-wise measures on  $D$  are proportional to the conditional measures with respect to  $\mathcal{A}$ . More precisely, let  $x(\nu_x^U|_{U_x})$  be the measure on  $X$  supported on  $xU_x$  such that for any Borel set  $V \subset U_x$  the value  $[x(\nu_x^U|_{U_x})](xV) = \nu_x^U(V)$ , then  $x(\nu_x^U|_{U_x})$  is proportional to  $\nu_x^{\mathcal{A}}$  for  $\nu$  almost every  $x \in D$ .

### 3. ENTROPY OF THE LIMIT MEASURE

**Theorem 3.1.** *Let  $\Gamma$  be a discrete subgroup of  $G = SL_n(\mathbb{R}) \times L$  where  $L$  is a product of semisimple algebraic groups. Let  $a \in G$  be an element from the class  $\mathcal{A}$  and let  $a_t = \text{diag}(e^t, \dots, e^t, e^{-(n-1)t}) \in SL_n(\mathbb{R})$  which commutes with  $a$ . Suppose the one-dimensional subgroup  $U$  is normalized by  $a$ ,  $a_t$  and contained in the unstable horospherical subgroup of  $a$ . Take  $\mu$  to be an  $a$ -invariant and ergodic probability measure on  $\Gamma \backslash G$  with  $h_{\mu}(a, U) > 0$ . If for a sequence of real numbers  $(T_n)$  with  $T_n \rightarrow \infty$  the limit*

$$\nu = \lim_{n \rightarrow \infty} \frac{1}{T_n} \int_0^{T_n} a_t \mu \, dt$$

*is none-zero, then  $h_{\nu}(a, U)(x) > 0$  for  $\nu$  almost every  $x$ .*

*Remark 3.2.* There is no need to assume  $U$  is one dimensional if  $a_t$  commutes with  $U$ . But in the case where  $U = U_1 \times U_2$ ,  $h_{\mu}(a, U_1) = h_{\mu}(a, U_2) = 0$  and  $a_t$  centralizes  $U_1$  but contracts or attracts  $U_2$ , the proof given below doesn't work.

We first do some preparation for the proof of Theorem 3.1 and establish some technical lemmas. To simplify the notation we denote  $G = G_0 \times G_1 \cdots \times G_l$  where  $G_0 = SL_n(\mathbb{R})$  and  $G_i$  is an algebraic group over  $k_i$  with a fixed norm  $|\cdot|_i$ .

We introduce a parameterization of neighborhoods for points in  $X$  using the Lie algebra of  $G$  so that we can construct sigma-rings of neighborhoods of  $X$  whose atoms are sections of  $U$  orbits. Let  $\mathfrak{g}_i$  be the Lie algebra of  $G_i$ , then the Lie algebra of  $G$  is  $\mathfrak{g} = \prod_{i=0}^l \mathfrak{g}_i$ . The exponential map  $\exp$  from  $\mathfrak{g}$  to  $G$  is defined by

$$(\dots, w_i, \dots) \in \mathfrak{g} \rightarrow (\dots, \exp(w_i), \dots) \in G.$$

Suppose  $U$  is a subgroup of  $G_j$  and the Lie algebra of  $U$  is  $\text{Lie}(U)$ . Let  $\tilde{\mathfrak{u}}$  be a  $k_j$ -linear subspace of  $\mathfrak{g}_j$  such that  $\text{Lie}(U) \oplus \tilde{\mathfrak{u}} = \mathfrak{g}_j$ . Take the linear space  $\mathfrak{v}$

of  $\mathfrak{g}$  to be the product of  $\mathfrak{g}_i$  except for the place  $j$  where it is the linear space  $\tilde{\mathfrak{u}}$ . For each factor of  $\mathfrak{v}$  we fix a basis and let  $\|\cdot\|_i$  be the corresponding sup norm induced by this basis and the norm on the base field  $k_i$ . We define a norm  $\|\cdot\|$  on  $\mathfrak{v}$  by

$$\|(w_0, \dots, w_l)\| = \sup_i (\|w\|_i).$$

Let  $V = \{\exp(v) : v \in \mathfrak{v}\}$ . The ball of radius  $r$  in  $V$  centered at  $g = \exp(w_0)$  with  $w_0 \in \mathfrak{v}$  is denoted by

$$B_r^V(g) = \{\exp(w_0 + w) : w \in \mathfrak{v}, \|w\|_{\mathfrak{v}} < r\}.$$

If  $g$  is the identity we simply denote this ball by  $B_r^V$ . As in Section 2, we take  $d^U$  to be the translation invariant metric on the unipotent subgroup  $U$  induced from the norm of the local field and  $B_r^U(g)$  or  $B_r^U$  if  $g$  is the identity is the ball of radius  $r$  centered at  $g \in U$ . For each  $x \in X$  there exists  $r > 0$  such that the map from  $B_r^V B_r^U = \{gh : g \in B_r^V, h \in B_r^U\}$  to  $X$  which sends  $vu$  to  $xvu$  is injective. We call such a number  $r$  an injectivity radius at  $x$ . For technical reasons we further require that injectivity radius  $r$  is small so that the product space  $B_r^V \times B_r^U$  can be naturally identified  $B_r^V B_r^U$ .

Let  $\nu$  be the non-zero limit measure in Theorem 3.1. Without loss of generality we assume that  $T_n = n$  and  $\nu = \lim \mu_n$  where

$$\mu_n = \frac{1}{n} \int_0^n a_t \mu \, dt.$$

Now we assume that the set of points  $x$  with  $h_\nu(a, U)(x) = 0$  has positive  $\nu$  measure and develop some consequences.

There exist  $\sigma > 0$ ,  $z \in X$  and  $r > 0$  such that  $2r$  is an injectivity radius at  $z$  and the set  $D = zB_r^V B_r^U$  has the following property:

$$(3.1) \quad \nu(\{x \in D : \nu_x^U \text{ is trivial}\}) > \sigma$$

Let  $D_2 = zB_r^V B_{2r}^U$ , we assume that the boundary of  $D_2$  in  $X$  has  $\nu$  measure zero. We are going to construct a countably generated sigma-ring on  $D_2$  subordinate to  $U$ .

**Lemma 3.3.** *There exist increasing sequences of finite partitions  $\mathcal{F}_m$  and  $\mathcal{C}_m$  ( $m \in \mathbb{N}$ ) of  $D_2$  such that the interiors of atoms of  $\mathcal{F}_m$  and  $\mathcal{C}_m$  have the form  $xB_s^V(h)B_r^U$  and  $xB_r^V B_s^U(g)$  respectively where  $s < \frac{1}{m}$ ,  $g \in B_r^U$  and  $h \in B_r^V$ . Furthermore the boundaries of atoms of  $\mathcal{F}_m$  and  $\mathcal{C}_m$  can be taken to have  $\nu$  measure zero.*

The proof of Lemma 3.3 uses the linearized parameterization of  $D_2$  introduced above. We omit the standard argument here. We fix increasing partitions  $\mathcal{F}_m$  and  $\mathcal{C}_m$  ( $m \in \mathbb{N}$ ) for  $D_2$  as in Lemma 3.3. Let  $\mathcal{F} = \bigvee_{m \in \mathbb{N}} \mathcal{F}_m$ , then it is a countably generated sigma-ring whose atoms are of the form  $yB_r^U$  with  $y \in D$ .

According to the property of the set  $D$  in (3.1) and the relationship between leaf-wise measures and conditional measures discussed in Section 2,

for any  $m > 0$ ,

$$(3.2) \quad \nu(\{x \in D : \nu_x^{\mathcal{F}}([x]_{\mathcal{C}_m}) = 1\}) > \sigma.$$

Now we want to reduce (3.2) to a similar equation about  $a_t\mu$  for some real number  $t$  using Johnson and Rudolph's argument in Section 5 of [8]. Fix some  $0 < \epsilon < \min(\sigma, \frac{1}{16})$  and  $m > 0$ , and set  $\mathcal{C} = \mathcal{C}_m$ . By (3.2) and the increasing martingale theorem, for  $N$  large enough and  $\mathcal{A} = \mathcal{F}_N$ , we have

$$\nu(\{x \in D : \nu_x^{\mathcal{A}}([x]_{\mathcal{C}}) > 1 - \epsilon/2\}) > \sigma - \epsilon/2.$$

Since atoms of  $\mathcal{A}$  and  $\mathcal{C}$  have  $\nu$  measure zero boundaries, there exists a positive integer  $n$  such that for

$$(3.3) \quad \mathcal{N} = \{x \in D : \mu_{n,x}^{\mathcal{A}}([x]_{\mathcal{C}}) > 1 - \epsilon\}$$

we have  $\mu_n(\mathcal{N}) > \sigma - \epsilon$ . Note that  $\mathcal{A}$  and  $\mathcal{C}$  are finite partitions and all the  $x$  in some atom  $[y]_{\mathcal{A} \vee \mathcal{C}}$  give the same value for  $\mu_{n,x}^{\mathcal{A}}([x]_{\mathcal{C}})$  which satisfies

$$\mu_n([x]_{\mathcal{A}})\mu_{n,x}^{\mathcal{A}}([x]_{\mathcal{C}}) = \mu_n([x]_{\mathcal{A} \vee \mathcal{C}}).$$

We remark here that  $\mathcal{N} \cap [x]_{\mathcal{A}}$  is either empty or an atom of  $\mathcal{A} \vee \mathcal{C}$  for any  $x \in D$  since  $\epsilon < \frac{1}{16}$ .

For each  $x \in \mathcal{N}$ , let

$$(3.4) \quad W(x) = \{0 \leq t \leq n : (a_t\mu)_x^{\mathcal{A}}([x]_{\mathcal{C}}) > 1 - \epsilon^{1/2}\}.$$

Let  $W(x)^c = [0, n] \setminus W(x)$ , then for any  $x \in \mathcal{N}$  and  $t \in W(x)^c$ ,

$$(3.5) \quad (a_t\mu)([x]_{\mathcal{A}} \setminus [x]_{\mathcal{A} \vee \mathcal{C}}) \geq \epsilon^{1/2}(a_t\mu)([x]_{\mathcal{A}}).$$

**Lemma 3.4.** *For any  $x \in \mathcal{N}$ , we have*

$$(3.6) \quad \frac{1}{n} \int_{W(x)} (a_t\mu)([x]_{\mathcal{A}}) dt > (1 - \epsilon^{1/2})\mu_n([x]_{\mathcal{A}}).$$

*Proof.* Assume the contrary. Then

$$\begin{aligned} \mu_n([x]_{\mathcal{A}} \setminus [x]_{\mathcal{A} \vee \mathcal{C}}) &\geq \frac{1}{n} \int_{W(x)^c} a_t\mu([x]_{\mathcal{A}} \setminus [x]_{\mathcal{A} \vee \mathcal{C}}) dt \\ (\text{by (3.5)}) &\geq \frac{1}{n} \int_{W(x)^c} \epsilon^{1/2}(a_t\mu)([x]_{\mathcal{A}}) dt \\ (\text{from the negation of (3.6)}) &\geq \epsilon\mu_n([x]_{\mathcal{A}}), \end{aligned}$$

which contradicts the definition of  $\mathcal{N}$  in (3.3).  $\square$

Next we set

$$(3.7) \quad A = \left\{0 \leq t \leq n : a_t\mu(\{x \in \mathcal{N} : t \in W(x)\}) > (1 - \epsilon^{1/4})a_t\mu(\mathcal{N})\right\}.$$

Then for any  $t \in A^c = [0, n] \setminus A$ , we have

$$(3.8) \quad a_t\mu(\{x \in \mathcal{N} : t \notin W(x)\}) \geq \epsilon^{1/4}a_t\mu(\mathcal{N}).$$

We remark here that for fixed  $t$ , the set  $\{x \in \mathcal{N} : t \notin W(x)\}$  is a union of atoms of  $\mathcal{A} \vee \mathcal{C}$  and there is at most one of them in each atom of  $\mathcal{A}$  since  $\mathcal{N} \cap [x]_{\mathcal{A}}$  contains at most one atom of  $\mathcal{A} \vee \mathcal{C}$ .

**Lemma 3.5.**

$$\frac{1}{n} \int_A a_t \mu(\mathcal{N}) dt > (1 - \epsilon^{1/4}) \mu_n(\mathcal{N}).$$

*Proof.* Assume the contrary. Let  $\mathcal{G} = \{(x, t) : x \in \mathcal{N}, t \notin W(x)\}$ . Since all the  $x \in [y]_{\mathcal{A} \vee \mathcal{C}}$  have the same set  $W(x)$ , we have that  $\mathcal{G}$  is a union of sets of the form  $[x]_{\mathcal{A} \vee \mathcal{C}} \times W(x)^c$ . Therefore by (3.8) we have

$$\begin{aligned} \frac{1}{n} \int_{A^c} a_t \mu(\mathcal{N}) dt &\leq \epsilon^{-1/4} \frac{1}{n} \int_{A^c} a_t \mu(\{x \in \mathcal{N} : t \notin W(x)\}) dt \\ &\leq \epsilon^{-1/4} \frac{1}{n} \sum_{[x]_{\mathcal{A} \vee \mathcal{C}} \subset \mathcal{N}} \int_{t \in W(x)^c} a_t \mu([x]_{\mathcal{A} \vee \mathcal{C}}) dt. \end{aligned}$$

Since  $\mathcal{N} \cap [x]_{\mathcal{A}}$  is either empty or a single atom of  $\mathcal{A} \vee \mathcal{C}$ , the right hand side of the last inequality is

$$\begin{aligned} &\leq \epsilon^{-1/4} \frac{1}{n} \sum_{[x]_{\mathcal{A} \vee \mathcal{C}} \subset \mathcal{N}} \int_{t \in W(x)^c} a_t \mu([x]_{\mathcal{A}}) dt \\ (\text{by Lemma 3.4}) \quad &< \epsilon^{1/4} \mu_n(\mathcal{N}). \end{aligned}$$

Now a simple calculation gives us the inequality of the lemma.  $\square$

In view of Lemma 3.5, there is some  $t \in A$  such that

$$a_t \mu(\mathcal{N}) \geq (1 - \epsilon^{1/4}) \mu_n(\mathcal{N}).$$

Now we fix such a number  $t$ . According to the definition of  $\mathcal{N}$ ,  $W(x)$  and  $A$  in (3.3), (3.4) and (3.7) respectively, we have

$$a_t \mu(\{x \in D : (a_t \mu)_x^{\mathcal{A}}([x]_{\mathcal{C}}) > 1 - \epsilon^{1/2}\}) \geq (1 - \epsilon^{1/4})^2 (\sigma - \epsilon).$$

Since  $((a_t \mu)_x^{\mathcal{A}})_y^{\mathcal{F}} = (a_t \mu)_y^{\mathcal{F}}$  for  $a_t \mu$  almost every  $x \in D_2$  and  $(a_t \mu)_x^{\mathcal{A}}$  almost every  $y \in D_2$ , we have

$$(3.9) \quad a_t \mu(\{x \in D : (a_t \mu)_x^{\mathcal{F}}([x]_{\mathcal{C}}) > 1 - \epsilon^{1/4}\}) \geq (1 - \epsilon^{1/4})^3 (\sigma - \epsilon).$$

It is not hard to see that  $(a_t \mu)_x^{\mathcal{F}} = \mu_{xa_{-t}}^{\mathcal{F}a_{-t}}$ . Recall that  $a_t$  normalizes the group  $U$  according to the assumption of Theorem 3.1, so the atoms of  $\mathcal{F}a_{-t}$  is still a section of  $U$  orbit. To get anything useful we need that  $a_{-t}$  does not change the shape of atoms of  $\mathcal{F}$  much which is the only obstruction for the more general case discussed in Remark 3.2. This happens if  $a_t$  commutes with  $U$ . If not we will use the property that  $\mu$  is invariant under the the diagonal element  $a$  to choose some integer  $p$  so that the right translation by the inverse of  $a_t a^p$  does not change the shape of atoms of  $\mathcal{F}$  much.

Since the right translation of  $a$  stretches foliations of  $U$ , there exists an integer  $p$  such that  $a_t a^p$  does not stretch  $U$  orbits, but  $a_t a^{p+1}$  does. Replace  $\mu$  in (3.9) by  $a^p \mu$  and rewrite it we get

$$(3.10) \quad \mu(\{x \in Da^{-p}a_{-t} : \mu_x^{\mathcal{F}a^{-p}a_{-t}}([x]_{\mathcal{C}a^{-p}a_{-t}}) > 1 - \epsilon^{1/4}\}) \geq (1 - \epsilon^{1/4})^3 (\sigma - \epsilon).$$



According to the construction of  $\mathcal{F}_m$  and  $\mathcal{C}_m$  in Lemma 3.3 there is a constant  $c \geq 2$  not depending on  $t$  such that

$$(3.11) \quad [x]_{\mathcal{F}a^{-p}a_{-t}} \vee \mathcal{C}a^{-p}a_{-t} \subset xB_{c/m}^U \quad \text{and} \quad xB_r^U \subset [x]_{\mathcal{F}a^{-p}a_{-t}}$$

for any  $x \in Da^{-p}a_{-t}$ . In view of (3.11), (3.10) and the relationship between conditional measures and leaf-wise measures in Section 2, if we set

$$D_m = \left\{ x \in X : \mu_x^U(B_{c/m}^U) > (1 - \epsilon^{1/4})\mu_x^U(B_r^U) \right\},$$

then

$$\mu(D_m) \geq (1 - \epsilon^{1/4})^3(\sigma - \epsilon).$$

It is easy to see that  $D_m \supset D_{m+1}$ . Since  $\epsilon$  is independent of  $m$ , we may let  $m$  go to infinity and conclude that  $\mu_x^U$  is trivial on a set whose  $\mu$  measure is strictly bigger than zero. This completes the proof of the following

**Lemma 3.6.** *Under the notation of Theorem 3.1, if the set of  $x$  with  $h_\nu(a, U)(x) = 0$  has positive  $\nu$  measure, then  $h_\mu(a, U)(x) = 0$  on a positive  $\mu$  measure set.*

*Proof of Theorem 3.1.* Suppose the contrary, then Lemma 3.6 implies that  $h_\mu(a, U)(x)$  is zero on a positive  $\mu$  measure set. Since  $\mu$  is ergodic under the map  $a$  and the function  $h_\mu(a, U)$  is  $a$ -invariant, we have  $h_\mu(a, U)(x) = 0$  for  $\mu$  almost every  $x$ . This contradicts the assumption for  $h_\mu(a, U)$ .  $\square$

#### 4. CONVERGENCE OF MEASURES ON QUOTIENTS OF $SL_2(\mathbb{R}) \times L$

In this section  $G = SL_2(\mathbb{R}) \times L$  where  $L$  is a product of semisimple algebraic groups over local fields of characteristic zero,  $H < G$  is the  $SL_2(\mathbb{R})$  factor of  $G$  and  $a_t = \text{diag}(e^t, e^{-t}) \in H$ . We will use Theorem 3.1 to prove some equidistribution results on homogeneous spaces using measure rigidity in Lindenstrauss [10]. We state a slightly variant version of Theorem 1.1 of [10] for the convenience of the reader.

**Theorem 4.1** ([10], Theorem 1.1). *Let  $\Gamma$  be a discrete subgroup of  $G$  such that  $\Gamma \cap L$  is finite. Suppose  $\nu$  is a probability measure on  $X = \Gamma \backslash G$ , invariant under the multiplication from the right by elements of the diagonal group flow  $a_t$ , assume that*

- (1) *All ergodic components of  $\mu$  with respect to the flow  $a_t$  have positive entropy.*
- (2)  *$\mu$  is  $L$ -recurrent.*

*Then  $\mu$  is a linear combination of algebraic measures invariant under  $H$ .*

**Corollary 4.2.** *Let  $U < H$  be a subgroup of the unstable horospherical subgroup of  $a = a_{t_0} \times c \in G$  where  $t_0 \neq 0$  and  $c \in L$  is a non-neutral class  $\mathcal{A}$  element. Let  $\Gamma$  be a discrete subgroup of  $G$  such that  $\Gamma \cap L$  is finite.*

Suppose  $\mu$  is an  $a$ -invariant and ergodic probability measure on  $\Gamma \backslash G$  such that  $h_\mu(a, U) > 0$ . If for a sequence of real numbers  $(T_n)$  with  $T_n \rightarrow \infty$

$$\nu = \lim_{n \rightarrow \infty} \frac{1}{T_n} \int_0^{T_n} a_t \mu \, dt,$$

then  $\nu$  is a linear combination of algebraic measures invariant under  $H$ .

*Proof.* Assume  $\nu$  is non-zero, otherwise there is nothing to prove. It is easy to see that  $\nu$  is invariant under the subgroup generated by  $a_t$  for  $t \in \mathbb{R}$  and  $c$ . Since  $c$  is from the class  $\mathcal{A}$  and non-neutral, the cyclic group generated by  $c$  is unbounded. It follows from the invariance of  $\nu$  under  $c$  that  $\nu$  is recurrent for the factor  $L$ . Note that the right translation of  $a_{t_0}$  also stretches  $U$  orbits. The conclusion of Theorem 3.1 implies that for  $\nu$  almost every  $x$ ,

$$h_\nu(a_{t_0}, U)(x) = h_\nu(a, U)(x) > 0.$$

So almost all the ergodic components of  $\nu$  with respect to the flow  $a_t$  have positive entropy. Now the assumptions of Theorem 1.1 of Lindenstrauss [10] are satisfied and the conclusion there is what we want.  $\square$

We say that two normal subgroups  $G_1$  and  $G_2$  of  $G$  are complementary if  $G = G_1 G_2$  and  $G_1 \cap G_2$  is finite. A lattice  $\Gamma$  of  $G$  is said to be irreducible if there are no proper complementary subgroups  $G_1$  and  $G_2$  of  $G$  such that  $(G_1 \cap \Gamma) \cdot (G_2 \cap \Gamma)$  has finite index in  $\Gamma$ . If  $L$  is a product of simply connected algebraic groups in the sense of [11] I.1.4.9 or connected real Lie groups, then the natural projection of an irreducible lattice  $\Gamma$  to  $L$  has dense image. We remark here that the group  $SL_m$  is simply connected.

**Lemma 4.3.** *Suppose  $\Gamma$  is a lattice of  $H \times L_1$  where  $L_1$  is a closed subgroup  $L$  and the natural projection of  $\Gamma$  to  $L_1$  has dense image. Then  $H$  acts uniquely ergodically on  $\Gamma \backslash H \times L_1$  with the probability Haar measure as the unique  $H$  invariant probability measure.*

The proof is standard and we omit it here.

**Corollary 4.4.** *Under the notation and assumption of Corollary 4.2, let  $L_1$  be the closure in  $L$  of the image of  $\Gamma$  under the natural projection map. Suppose in addition that  $\Gamma$  is an irreducible lattice and  $\mu$  is supported on  $X = \Gamma \backslash H \times L_1$ , then  $\nu = \nu(X)m_X$ . Moreover if  $\mu$  has non-escape of mass for the flow  $a_t$ , then*

$$(4.1) \quad \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T a_t \mu \, dt = m_X.$$

*Proof.* Corollary 4.2 implies that  $\nu$  is invariant under the first factor  $H$  of  $G$ . Since  $X$  is closed and  $\mu$  is supported on  $X$ , the measure  $\nu$  is supported on  $X$ . Since the natural projection of  $\Gamma$  to  $L_1$  has dense image, Lemma 4.3 implies that  $\nu = \nu(X)m_X$ .

If  $\mu$  has non-escape of mass, then  $\nu = m_X$ . Since the sequence  $(T_n)$  is arbitrary, (4.1) follows.  $\square$

**Example 4.5.** Let  $k$  be a totally real quadratic number field with ring of integers  $\mathfrak{o}$ . Let  $\tau$  be the non-trivial Galois automorphism of  $k$  over  $\mathbb{Q}$ . For each  $g \in SL_2(\mathfrak{o})$  let  $g^\tau$  be the componentwise conjugation, then  $SL_2(\mathfrak{o})$  embeds in  $G = SL_2(\mathbb{R}) \times SL_2(\mathbb{R})$  through the map  $g \rightarrow (g, g^\tau)$  as an irreducible lattice. Suppose  $\gamma \in SL_2(\mathfrak{o})$  is a diagonal matrix whose eigenvalues are not root of unity.

Take  $X = SL_2(\mathfrak{o}) \backslash G$  and  $\vartheta_X = SL_2(\mathfrak{o}) \in X$ . It is not hard to see that the periodic orbit  $\vartheta_X \begin{pmatrix} 1 & 0 \\ \mathbb{R}^2 & 1 \end{pmatrix}$  is invariant under  $\gamma$ . Moreover the action of  $\gamma$  on this periodic orbit is isomorphic to a hyperbolic action on  $\mathbb{T}^2$ . We claim that if we put a positive entropy ergodic measure of  $\mathbb{T}^2$  on this periodic orbit to get a measure  $\mu$ , then  $\mu$  has positive entropy contribution along some one-dimensional unipotent orbit. The idea is that there are two directions for this  $\mathbb{R}^2$  orbit and one of them is stretched by  $\gamma$  and the other is contracted.

If  $L = SL_2(\mathbb{R})$ , then we could give a partial answer to Conjecture 9.2 of Einsiedler [1].

**Corollary 4.6.** *Let  $\mu$  be an ergodic probability measure on the irreducible quotient of  $\Gamma \backslash SL_2(\mathbb{R}) \times SL_2(\mathbb{R})$  with respect to the group generated by*

$$b_t = \text{Id} \times \text{diag}(e^t, e^{-t}) \quad t \in \mathbb{R}.$$

*If  $\mu$  has positive entropy with respect to  $b_t$  when  $t \neq 0$  and has non-escape of mass for the flow  $a_t$ , then*

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T a_t \mu \, dt = m_{\Gamma \backslash G}.$$

The proof is similar to that of Corollary 4.4 but uses a variant of Theorem 3.1. In above corollary the measure  $\mu$  is ergodic under the flow  $b_t$ , whereas in Theorem 3.1 the given measure  $\mu$  is invariant under an invertible map. It is not hard to see that the proof of Theorem 3.1 still works in this case to give positive entropy of ergodic components of the limit measure.

## 5. CONVERGENCE OF MEASURES ON $SL_2(\mathbb{Z}) \backslash SL_2(\mathbb{R})$

Let  $n > 1$  be a positive integer and let  $\lambda$  be a probability measure on  $[0, 1] = \mathbb{Z} \backslash \mathbb{R}$  such that  $\lambda$  is  $\times n$ -invariant and ergodic with positive entropy. Recall that for  $s \in \mathbb{R}$ ,

$$u(s) = \begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix} \in SL_2(\mathbb{R}).$$

Let  $Y = SL_2(\mathbb{Z}) \backslash SL_2(\mathbb{R})$  and  $\vartheta_Y = SL_2(\mathbb{Z}) \in Y$ . Let  $\tilde{\mu}$  be the measure on  $Y$  such that for any  $f \in C_c(Y)$

$$(5.1) \quad \int_Y f \, d\tilde{\mu} = \int_0^1 f(\vartheta_Y u(s)) \, d\lambda(s).$$

We also use  $\times n$  to denote the measure preserving map on  $(Y, \tilde{\mu})$  that sends  $\vartheta_Y u(s)$  to  $\vartheta_Y u(ns)$ .

To guarantee that there is non-escape of mass under diagonal flow  $a_t = (e^t, e^{-t})$  we need some additional assumption on the measure  $\lambda$ . One possible choice is to assume that  $\lambda$  is friendly, which is defined in [9]. Examples of friendly measures consist of Hausdorff measures supported on self-similar sets such as Cantor sets. The following theorem is a general version of Theorem 1.1.

**Theorem 5.1.** *Let  $\lambda$  be friendly probability measure on  $[0, 1]$ . Suppose that  $\lambda$  is  $\times n$ -invariant and ergodic with positive entropy. Let  $\tilde{\mu}$  be the measure on  $Y$  induced from  $\lambda$  by (5.1), then*

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T a_t \tilde{\mu} dt = m_Y.$$

To prove Theorem 5.1, we need first prove Theorem 1.2. For a field  $k$  of characteristic zero, let  $PGL_2(k)$  be the  $k$  points of the algebraic group  $PGL_2$ . The natural map  $\pi_k : GL_2(k) \rightarrow PGL_2(k)$  is surjective and the kernel consists of diagonal matrices of  $GL_2(k)$ . For a ring  $R \subset k$ , we use  $PGL_2(R)$  to denote the image of  $GL_2(R)$  under  $\pi_k$ .

Suppose the prime decomposition of  $n = p_1^{\sigma_1} \cdots p_l^{\sigma_l}$ . Let

$$S = \{\infty, p_1, \dots, p_l\}, \quad \mathbb{Q}_S = \prod_{v \in S} \mathbb{Q}_v, \quad K = \prod_{p \in S, p \neq \infty} PGL_2(\mathbb{Z}_p).$$

Take  $\Gamma = PGL_2(\mathbb{Z}[\frac{1}{p_1}, \dots, \frac{1}{p_l}])$  which sits diagonally in the group  $G = PGL_2(\mathbb{Q}_S)$  as an irreducible lattice. It is not hard to see that

$$X = \Gamma \backslash G = PGL_2(\mathbb{Z}) \backslash PGL_2(\mathbb{R}) \times K.$$

We can define a factor map  $\eta : X \rightarrow Y$  which sends  $(g, h) \bmod \Gamma$  in  $X$  with  $g \in PGL_2(\mathbb{R})$  and  $h \in K$  to  $g \bmod SL_2(\mathbb{Z})$  in  $Y$ . For  $\mathbf{v} \in \mathbb{Q}_S$ , let  $u(\mathbf{v}) = \begin{pmatrix} 1 & 0 \\ \mathbf{v} & 1 \end{pmatrix}$ . Let  $V$  be the unipotent subgroup of  $G$  generated by  $u(\mathbf{v})$  for  $\mathbf{v} \in \mathbb{Q}_S$ .

**Lemma 5.2.** *Let  $\mu$  be an  $a$ -invariant and ergodic probability measure on  $X$  with positive entropy. Suppose  $\mu$  is supported on  $\vartheta_X u(\mathbb{Q}_S)$ , then  $h_\mu(a, U) > 0$  where  $U = \{u(s) : s \in \mathbb{R}\}$ .*

*Proof.* Let  $\Lambda = V \cap \Gamma$ , then  $\Lambda$  is a lattice of  $V$  and  $\Gamma V \cong \Lambda \backslash V$  is an  $a$ -invariant compact subset of  $X$ . The measure  $\mu$  can be viewed as a probability measure on  $\Lambda \backslash V$ . Note that there is only one expanding foliations for  $a$  on  $\Lambda \backslash V$ , namely, the orbits of  $U$ . Since  $h_\mu(a) > 0$  and  $\mu$  is ergodic under the map  $a$ , we have  $h_\mu(a, U) > 0$ .  $\square$

*Proof of Theorem 1.2.* Let  $L = \prod_{v \in S, v \neq \infty} SL_2(\mathbb{Q}_v)$ . It is not hard to see that there is an irreducible lattice  $\Gamma_0$  of  $G_0 = SL_2(\mathbb{R}) \times L$  such that  $X \cong X_0 = \Gamma_0 \backslash G_0$  under the natural map  $G_0 \rightarrow G$ . Let  $L_1$  be the closure in  $L$  of the image of  $\Gamma_0$  under the natural projection map. Let  $b = \text{diag}(\sqrt{n}, 1/\sqrt{n}) \times$

$\text{diag}(n, 1)^l \in G_0$  and let  $\mu_0$  be the measure on  $X_0$  correspond to  $\mu$  on  $X$ . Then the system  $(X, \mu, a, a_t)$  is isomorphic to the system  $(X_0, \mu_0, b, a_t)$ .

Note that  $\mu_0$  is supported on  $\Gamma_0(SL_2(\mathbb{R}) \times L_1)$ . The assumption that  $h_\mu(a) > 0$  implies  $h_{\mu_0}(b, U) > 0$  according to Lemma 5.2. Since  $\mu$  has non-escape of mass for the flow  $a_t$ , the conclusion follows from Corollary 4.4.  $\square$

Now we return to the proof of Theorem 5.1. The measure  $\lambda$  on  $[0, 1]$  induces a measure  $\mu_1$  on  $X$  such that for any  $f \in C_c(X)$ ,

$$\int_X f \, d\mu_1 = \int_0^1 f(\vartheta_X u(s)) \, d\lambda(s).$$

Then the pushforward  $\eta\mu_1 = \tilde{\mu}$ . Note that  $\mu_1$  is not invariant for the right translation of  $a = \text{diag}(n, 1)^{l+1} \in G$ . But we can still guarantee that there is an  $a$ -invariant and ergodic measure on  $X$  that projects to  $\tilde{\mu}$ .

**Lemma 5.3.** *There exists an  $a$ -invariant and ergodic probability measure  $\mu$  on  $X$  supported on  $\vartheta_X u(\mathbb{Q}_S)$  where  $\vartheta_X = \Gamma \in X$  such that  $\eta\mu = \tilde{\mu}$  and  $h_\mu(a) > 0$ .*

*Proof.* Let  $\mu_1$  be as above and let  $\mu_2$  be a limit point in the weak\* topology for the sequence  $\frac{1}{N} \sum_{i=0}^{N-1} a^i \mu_1$ . It is clear that  $\mu_2$  is  $a$ -invariant. We claim that  $\mu_2$  is a probability measure. Since  $\Lambda = V \cap \Gamma$  is a lattice of  $V$ . It follows that  $\Gamma V \cong \Lambda \backslash V$  is an  $a$ -invariant compact subset of  $X$ . Since each  $a^i \mu_1$  is supported on  $\Gamma V$ , the measure  $\mu_2$  is a probability measure.

Now  $\eta$  is a factor map from  $(X, \mu_2, a)$  to  $(Y, \tilde{\mu}, \times n)$  and  $h_{\tilde{\mu}}(\times n) > 0$ , so  $h_{\mu_2}(a) > 0$ . Take  $\mu$  to be some ergodic component of  $\mu_2$  with positive entropy, then  $\mu$  satisfies the requirement of the lemma.  $\square$

*Proof of Theorem 5.1.* By Lemma 5.3, there exists an  $a$ -invariant and ergodic probability measure  $\mu$  on  $X$  such that  $\eta\mu = \tilde{\mu}$  and  $h_\mu(a) > 0$ . Moreover, the following diagram

$$\begin{array}{ccc} (X, \mu) & \xrightarrow{a_t} & (X, \mu) \\ \eta \downarrow & & \downarrow \eta \\ (Y, \tilde{\mu}) & \xrightarrow{a_t} & (Y, \tilde{\mu}) \end{array}$$

commutes. Since  $\lambda$  is friendly, there is non-escape of mass for  $\tilde{\mu}$  under the flow  $a_t$ . The commutative diagram above and the fact that fibers of  $\eta$  are compact imply that there is non-escape of mass for  $\mu$  under the flow  $a_t$ . Therefore  $\mu$  satisfies all the assumptions of Theorem 1.2. So  $\mu$  is equidistributed on average for the flow  $a_t$  with respect to an  $H$  invariant homogeneous measure. Therefore  $\tilde{\mu}$  is equidistributed on average for the flow  $a_t$  on  $Y$ .  $\square$

## 6. CONVERGENCE OF MEASURES ON $SL_3(\mathbb{Z}) \backslash SL_3(\mathbb{R})$

In the next two sections let  $G = SL_3(\mathbb{R})$ ,  $\Gamma = SL_3(\mathbb{Z})$  and  $Z = \Gamma \backslash G$ . Other notations are the same as in Theorem 1.3 which we prove right now. Let  $\nu = \lim \mu_n$  and we want to show that  $\nu$  is a scalar multiple of the probability Haar measure. Without loss of generality we assume that  $\nu \neq 0$ .

For the hyperbolic matrix  $\gamma \in SL_2(\mathbb{Z})$ , let  $g_\gamma = \begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix}$ . Then  $\mu$  is  $g_\gamma$ -invariant and has positive entropy. Similar to Lemma 5.2, we have

**Lemma 6.1.** *For the probability measure  $\mu$  in Theorem 1.3, then there exists a one dimensional subgroup of the unstable horospherical subgroup of  $g_\gamma$  such that  $h_\mu(g_\gamma, U)(x) > 0$  for  $\mu$  almost every  $x$ .*

It follows from Theorem 3.1 that the average measure  $\nu$  in Theorem 1.3 has positive entropy contribution along  $U$  orbits almost surely. So a typical ergodic component of  $\nu$  with respect to the group  $\Lambda$  generated by  $a_t$  and  $g_\gamma$ , say probability measure  $\nu_1$ , satisfies  $h_{\nu_1}(g_\gamma) > 0$ .

Let  $A$  be the Cartan subgroup of  $G$  containing  $\Lambda$ , then

$$(6.1) \quad \nu_2 = \int_{\Lambda \backslash A} a \nu_1 \, dm_{\Lambda \backslash A}(a)$$

is an  $A$ -invariant probability measure with all the  $A$ -ergodic components having positive entropy with respect to  $g_\gamma$ . Now we need to use the measure rigidity theorem of Einsiedler, Katok and Lindenstrauss [4].

**Theorem 6.2** ([4]). *Let  $\mu$  be an  $A$ -invariant and ergodic probability measure on  $Z$ , Assume that there is some one-parameter subgroup of  $A$  which acts on  $Z$  with positive entropy. Then  $\mu$  is the unique  $SL_3(\mathbb{R})$  invariant measure.*

In view of (6.1) and the positive entropy of  $\nu_1$  with respect to  $g_\gamma$ , we have that all the ergodic component of  $\nu_2$  with respect to the Cartan subgroup  $A$  have positive entropy for the map  $g_\gamma$ . Theorem 6.2 implies that  $\nu_2 = m_Z$ . By (6.1),

$$\int_{\Lambda \backslash A} h_{a\nu_1}(g_\gamma) \, dm_{\Lambda \backslash A}(a) = h_{m_Z}(g_\gamma).$$

Since the group  $A$  is abelian, we have  $h_{a\nu_1}(g_\gamma) = h_{\nu_1}(g_\gamma)$ . Therefore,  $h_{\nu_1}(g_\gamma) = h_{m_Z}(g_\gamma)$ .

According to the results in Section 9 of [12],  $\nu_1$  and  $m_Z$  have the same entropy if and only if they are equal. Since  $\nu_1$  is typical in the ergodic decomposition of  $\nu$ , the measure  $\mu$  must be a scalar multiple of  $m_Z$ . This completes the proof of Theorem 1.3.

**Corollary 6.3.** *Under the notation and assumption of Theorem 1.3, if  $\mu$  has non-escape of mass with respect to the flow  $a_t$ , then*

$$(6.2) \quad \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T a_t \mu \, dt = m_Z.$$

## 7. APPLICATIONS TO DIOPHANTINE APPROXIMATION

Let  $\|\cdot\|$  be the sup norm on the Euclidean space  $\mathbb{R}^n$ . We say that  $\mathbf{v} \in \mathbb{R}^2$  is Dirichlet improvable (or DI for short), if there exists  $0 < \sigma < 1$  such that for sufficiently large integer  $N$ , there are integer  $n$  and vector  $\mathbf{w}$  such that

$$(7.1) \quad \|n\mathbf{v} - \mathbf{w}\| < \sigma N^{-1} \quad \text{and} \quad 0 < |n| < \sigma N^2.$$

In this case we say that  $\mathbf{v}$  can be  $\sigma$ -improved. Let

$$DI_\sigma = \{\mathbf{v} \in \mathbb{R}^2 : \mathbf{v} \text{ can be } \sigma\text{-improved.}\} \quad \text{and} \quad DI = \bigcup_{0 < \sigma < 1} DI_\sigma.$$

Let the notation be as in Section 6. Let  $\pi : G \rightarrow Z$  be the natural projection. The dynamical approach to this Diophantine approximation problem is through the map

$$u : \mathbb{R}^2 \rightarrow G \quad \text{where} \quad u(\mathbf{v}) = \begin{pmatrix} \text{Id}_2 & 0 \\ \mathbf{v} & 1 \end{pmatrix},$$

then study the orbit of  $\pi(u(\mathbf{v}))$ . Recall that  $Z$  can be identified with the set of unimodular lattices of  $\mathbb{R}^3$  and the correspondence is  $\pi(g) \rightarrow \mathbb{Z}^3 g$  for  $g \in G$ . Let  $K_\sigma$  be the set of unimodular lattices in  $\mathbb{R}^3$  whose shortest nonzero vector has norm bigger than or equal to  $\sigma$ . Then  $\mathbf{v} \in DI_\sigma$  implies that for  $T$  large enough  $\pi(u(\mathbf{v}))a_t \notin K_{\sqrt{\sigma}}$ .

**Theorem 7.1.** *Let  $\gamma \in SL_2(\mathbb{Z})$  be a hyperbolic matrix and let  $\lambda$  be a  $\gamma$ -invariant and ergodic probability measure on  $[0, 1]^2$ . Let  $\mu = (\pi u)\lambda$  and  $a_t = (e^t, e^t, e^{-2t})$ . If  $\lim_{T \rightarrow \infty} \int_0^T a_t \mu \, dt \neq 0$ , then Dirichlet's theorem is not improvable  $\lambda$  almost surely.*

*Proof.* Assume that  $\lambda(DI) > 0$ , then for some  $0 < \sigma < 1$  we have  $\lambda(DI_\sigma) = \tau > 0$ . Let  $(T_n)$  be a sequence of real numbers such that  $T_n \rightarrow \infty$  and

$$\lim_{n \rightarrow \infty} \frac{1}{T_n} \int_0^{T_n} (a_t \pi u) \lambda \, dt = \nu.$$

By passing to a subsequence, we may assume that

$$\lim_{n \rightarrow \infty} \frac{1}{T_n} \int_0^{T_n} (a_t \pi u) \lambda|_{DI_\sigma} \, dt = \nu_1.$$

Since elements of  $\pi(u(DI_\sigma))$  eventually do not intersect  $K_{\sqrt{\sigma}}$  under the flow  $a_t$ , we have  $\nu_1(K_{\sqrt{\sigma}}) = 0$ . We remark here that  $K_{\sqrt{\sigma}}$  is a neighborhood of  $\mathbb{Z}^3$ .

Take  $g_\gamma = \begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix} \in G$ , then

$$\lim_{n \rightarrow \infty} \frac{1}{T_n} \int_0^{T_n} (a_t \pi u) \lambda|_{\gamma^{-1}(DI_\sigma)} \, dt = g_\gamma \nu_1.$$

Since  $m_Z$  is ergodic for the flow  $a_t$ , it is the only possible ergodic component of  $\nu$  for the flow  $a_t$  according to Theorem 1.3. Note that  $g_\gamma \nu_1$  is invariant for the flow  $a_t$  and it vanishes on some open subset. Therefore  $g_\gamma \nu_1 = 0$

Since  $\lambda$  is  $\gamma$ -ergodic, we see that  $\nu = 0$ . Since the sequence  $(T_n)$  is arbitrary, this implies  $\lim_{T \rightarrow \infty} \int_0^T a_t \mu dt = 0$  which contradicts to the assumption. Therefore  $\lambda(DI) = 0$ .  $\square$

Now we give some examples of  $\lambda$  such that there is some mass left for the flow  $a_t$  and prove Theorem 1.4. Let  $B_r(\mathbf{v})$  with  $\mathbf{v} \in \mathbb{R}^2$  be the ball of radius  $r$  centered at  $\mathbf{v}$  under the norm  $\|\cdot\|$ . We need the following well-known result:

**Proposition 7.2.** *Let  $\gamma \in SL_2(\mathbb{Z})$  be a hyperbolic matrix. Suppose  $\lambda$  is a  $\gamma$ -invariant and ergodic probability measure on  $\mathbb{T}^2$  such that  $h_\lambda(\gamma) = c h_m(\gamma)$  where  $m$  is the Lebesgue measure and  $0 \leq c \leq 1$ , then  $\lambda$  has exact dimension  $2c$ , that is*

$$\lim_{\epsilon \rightarrow 0} \frac{\log \mu(B_\epsilon(\mathbf{v}))}{\log \epsilon} = 2c$$

for  $\lambda$  almost every  $\mathbf{v} \in \mathbb{R}^2$ .

Let  $G^+$  and  $G^-$  be the unstable and stable horospherical subgroup of  $a_1$  in  $G$  and let  $C$  be the centralizer of  $a_1$ . We fix a right invariant Riemannian metric on  $G^+$  and  $G^-C$  respectively, and denote by  $B_r^{G^+}$  and  $B_r^{G^-C}$  the ball of radius  $r$  centered at the identity with respect to the induced metric.

**Definition 7.3.** For a probability measure  $\mu$  on  $Z$  we say that  $\mu$  has dimension at least  $d$  in the unstable direction of  $a_1$  if for any  $\delta > 0$  there exists  $\kappa > 0$  such that for any  $\epsilon \in (0, \kappa)$  and for any  $\sigma \in (0, \kappa)$  we have

$$\mu(xB_\epsilon^{G^+} B_\sigma^{G^-C}) \ll_\delta \epsilon^{d-\delta} \quad \text{for any } x \in Z.$$

Let  $\lambda$  be as in Proposition 7.2, then  $\mu = (\pi u)\lambda$  has at least dimension  $2c$  in the unstable direction of  $a_1$ . If  $c > \frac{2}{3}$ , then Theorem 1.6 of Einsiedler and Kadyrov [3] shows that there is some mass left.

**Theorem 7.4** ([3]). *For a fixed  $d$ , let  $\mu$  be a probability measure on  $Z$  of dimension at least  $d$  in the unstable direction. Let  $(T_n)$  be a sequence of positive real numbers such that*

$$\nu = \lim_{n \rightarrow \infty} \int_0^{T_n} a_t \mu dt.$$

Then  $\nu(Z) \geq \frac{3}{2}(d - \frac{4}{3})$ .

Now Theorem 1.4 follows from Theorem 7.1, Proposition 7.2 and Theorem 7.4.

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